Int J Theor Phys (2010) 49: 2841–2851
DOI 10.1007/s10773-010-0476-1

Generation of Genuine Tripartite Macroscopic Entanglement in Y-type System

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Received: 31 March 2010 / Accepted: 19 August 2010 / Published online: 7 September 2010

Abstract In this paper, we study the generation of genuine tripartite entanglement in a Y-type system. Employing standard method of laser theory, we deduce the dynamic evolution equation of the three-mode field. It is showed that genuine tripartite macroscopic entanglement can be produced in our scheme.

Keywords Genuine tripartite entanglement · Y-type atom

1 Introduction

Quantum entanglement is a very important research topic in quantum computation and quantum communication [1]. Generating entangled states is always a valuable research in quantum information science. In the past decades, entangled states between individual qubits have been extensively studied where entanglement is carried by discontinuous states. In recent years, continuous-variable (CV) entanglement has attracted much attention [1–19] because its many advantages [1]. Traditional schemes of generating CV entanglement is based on the parametric down conversion process [20, 21]. But, since Xiong et al. [9] proposed a scheme to generate two-mode macroscopic entangled lights based on correlated spontaneous emission lasers (CSEL), many other schemes based on CSEL system are proposed. For example, Tan et al. [10, 11] proposed a similar scheme and showed that a macroscopic entangled state between two modes of the radiation field can be built in a cavity. Kiffner [12] proposed a scheme for generating two-mode entanglement in macroscopic light just using a single atom. Other schemes to produce macroscopic entangled light are presented in [14, 18].

However, only the two-mode entanglement is considered in those schemes [9–12]. Multipartite CV entanglement, as a kind of entanglement resource which receives a lot of attentions [2–8], seems more valuable to be studied for its various applications. For example, it

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enables one to construct a quantum teleportation network [3], to build an optimal one-to-two telecloner [4], or to perform controlled dense coding [5]. In fact, tripartite entanglement for CV has been generated in experiment by using three squeezed vacuum states and two beam splitters [6]. Recently, Cassemiro and Villar demonstrated that scalable multipartite entanglement among light fields may be generated by optical parametric oscillators [8]. Moreover, some schemes of generating tripartite entanglement with OPOs have been also proposed [22–24], and a most recent research showed that tripartite entanglement also can be generated in cavity -QED system [25].

In this paper, we propose a scheme to generate three-mode entangled light fields in a cavity via the interaction between Y-type atoms and the cavity. We deduce the master equation of the three-mode fields. It is noteworthy that when it comes to tripartite entanglement, genuine tripartite entanglement which makes it different from bipartite entanglement should be paid enough attention. We discuss genuine tripartite entanglement condition carefully. By employing the PPT criterion [26], all the bipartite entanglement is proved. In addition, we have also proved the feasibility of generating genuine tripartite entanglement within our system and show the existence of the free entanglement [27].

2 The Model and Calculation

We consider an experimental situation in which a series of atoms with four levels are injected into a cavity at a constant rate r_a. The level configuration of the atoms is depicted in Fig. 1. The four levels of the Y-type atoms are denoted by $|a\rangle$, $|b\rangle$, $|c\rangle$ and $|d\rangle$, and the three cavity modes resonantly interact with the atomic transitions $|a\rangle \longleftrightarrow |c\rangle$, $|b\rangle \longleftrightarrow |c\rangle$, and $|c\rangle \longleftrightarrow |d\rangle$ with coupling constants g_1, g_2 and g_3, respectively. The frequencies of the three modes are ω_{ac}, ω_{bc} and ω_{cd}. This configuration of the Y-type atomic levels has been studied extensively both theoretically [28–30] and experimentally [31]. Based on the previous experiment [31], atomic sodium can be selected as the Y-type atoms, and four levels $5S_{1/2}, 4D_{3/2}, 3P_{3/2}, 3S_{1/2}$ can be considered as the four atomic levels $|a\rangle$, $|b\rangle$, $|c\rangle$ and $|d\rangle$. In our scheme, two strong classical fields drive the atomic level resonantly between $|a\rangle \leftrightarrow |d\rangle$ and $|b\rangle \leftrightarrow |d\rangle$ with Rabi frequencies Ω_1 and Ω_2 (through two-photon transition). In the interaction picture, the Hamiltonian of the system can be expressed as

$$\begin{aligned} H = {} & g_1 a_1 |a\rangle\langle c| + g_2 a_2 |b\rangle\langle c| + g_3 a_3 |c\rangle\langle d| \\ & + \Omega_1 e^{-i\varphi_1} |a\rangle\langle d| + \Omega_2 e^{-i\varphi_2} |b\rangle\langle d| + \varepsilon a_1^\dagger a_2^\dagger + h.c., \end{aligned} \tag{1}$$

where a_j ($a_j^\dagger$) is the annihilation (creation) operator of the cavity mode j, g_j is the coupling constant, and Ω_j is a real number. Unlike our previous work [25], an optical parametric

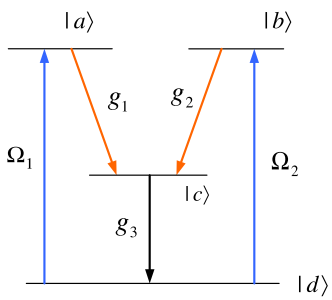

Fig. 1 Level configuration of atoms. Three cavity modes resonantly interact with atomic transitions $|a\rangle \leftrightarrow |c\rangle$, $|b\rangle \leftrightarrow |c\rangle$, and $|c\rangle \leftrightarrow |d\rangle$ with coupling constants g_1, g_2, and g_3, respectively, and the frequencies of the three modes are ω_{ac}, ω_{bc} and ω_{cd}. Two classical fields drive the atomic level resonantly between $|a\rangle \leftrightarrow |d\rangle$ and $|b\rangle \leftrightarrow |d\rangle$ with Rabi frequencies Ω_1 and Ω_2

oscillator (OPO) with strength ε is introduced into the cavity which can convert a pump field into two cavity mode 1 (with frequency ω_{ac}) and 2 (with frequency ω_{bc}).

We would like to make a brief analysis about how the entanglement is generated in this model, and give the precise calculation later. As it is shown in Fig. 1, $g_2 = 0$, $\Omega_2 = 0$, then our scheme can be reduced into a three level cascade model which has been discussed in Ref. [9] and Ref. [10, 11]. According to Xiong's work [9], mode 1 and mode 3 should be entangled. Again, if the atomic level $|a\rangle$ is absent, mode 2 and mode 3 should be entangled. In fact, the entanglement generated in cascade system results from the fact that two photons must be produced in one cycle. One photon originates from the transaction $|a\rangle \leftrightarrow |c\rangle$ ($|b\rangle \leftrightarrow |c\rangle$), and the other photon originates from the transaction $|c\rangle \leftrightarrow |d\rangle$. Including an optical parametric oscillator with mode 1 and mode 2, when both $|a\rangle$ and $|b\rangle$ present, the tripartite should be entangled.

Using the method of linear laser theory [32], we derive the master equation for multi-mode field. When tracing over the atomic states is taken, the reduced density operator of the field is written as

$$\begin{aligned}\dot{\rho}_f &= -\frac{i}{\hbar}Tr_a([H,\rho])\\ &= -\frac{i}{\hbar}\{[H_{ac},\rho_{ca}] + [H_{bc},\rho_{cb}] + [H_{cd},\rho_{dc}]\\ &\quad + \varepsilon(a_1^\dagger a_2^\dagger \rho - \rho a_1^\dagger a_2^\dagger) + h.c.\},\end{aligned} \tag{2}$$

where the operator ρ_f stands for the density operator of the cavity field. For simplicity, we assume that the atomic levels decay with the same rate γ. The matrix elements of the density operators ρ_{ca} etc. can be obtained by solving the following matrix equation

$$\frac{d}{dt}R = -MR + D, \tag{3}$$

where

$$\begin{aligned}M &= \begin{bmatrix}\gamma & 0 & -i\tilde{\Omega}_1\\ 0 & \gamma & -i\tilde{\Omega}_2\\ -i\tilde{\Omega}_1^* & -i\tilde{\Omega}_2^* & \gamma\end{bmatrix}, \quad \text{with } \tilde{\Omega}_i = \Omega_i e^{i\varphi_i}\,(i=1,2)\\ D &= -i\begin{bmatrix} g_1 a_1^\dagger \rho_{aa} + g_2 a_2^\dagger \rho_{ba} - g_3 a_3 \rho_{da} - \rho_{cc} a_1^\dagger\\ g_1 a_1^\dagger \rho_{ab} + g_2 a_2^\dagger \rho_{bb} - g_3 a_3 \rho_{db} - \rho_{cc} a_2^\dagger\\ g_1 a_1^\dagger \rho_{ad} + g_2 a_2^\dagger \rho_{bd} - g_3 a_3 \rho_{dd} - \rho_{cc} a_3^\dagger\end{bmatrix},\\ R &= \begin{bmatrix}\rho_{ca}\\ \rho_{cb}\\ \rho_{cd}\end{bmatrix}.\end{aligned} \tag{4}$$

The linear solution in coupling constant g_i is given by $R = M^{-1}D$. Here, we use A_{ij} to represent the elements of the matrix M^{-1}, that is

$$A = M^{-1} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}. \tag{5}$$

Now, we just consider classical fields to determine the zero order approximation of the density matrix elements ρ_{aa}, ρ_{ca}, and ρ_{cc} in the vector D from the following equation

$$\frac{d}{dt}\widetilde{R} = -\widetilde{M}\widetilde{R} + \widetilde{D}, \tag{6}$$

where

$$\widetilde{M} = \begin{bmatrix} \gamma & 0 & 0 & -i\tilde{\Omega}_1 & i\tilde{\Omega}_1^* & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & -i\tilde{\Omega}_2 & i\tilde{\Omega}_2^* & 0 & 0 \\ 0 & 0 & \gamma & i\tilde{\Omega}_1 & -i\tilde{\Omega}_1^* & i\tilde{\Omega}_2 & -i\tilde{\Omega}_2^* & 0 & 0 \\ -i\tilde{\Omega}_1^* & 0 & i\tilde{\Omega}_1^* & \gamma & 0 & 0 & 0 & -i\tilde{\Omega}_2^* & 0 \\ i\tilde{\Omega}_1 & 0 & -i\tilde{\Omega}_1 & 0 & \gamma & 0 & 0 & 0 & i\tilde{\Omega}_2 \\ 0 & -i\tilde{\Omega}_2^* & i\tilde{\Omega}_2^* & 0 & 0 & \gamma & 0 & 0 & -i\tilde{\Omega}_1^* \\ 0 & i\tilde{\Omega}_2 & -i\tilde{\Omega}_2 & 0 & 0 & 0 & \gamma & i\tilde{\Omega}_1 & 0 \\ 0 & 0 & 0 & -i\tilde{\Omega}_2 & 0 & 0 & i\tilde{\Omega}_1 & \gamma & 0 \\ 0 & 0 & 0 & 0 & i\tilde{\Omega}_2^* & -i\tilde{\Omega}_1 & 0 & 0 & \gamma \end{bmatrix},$$

$$\tilde{R} = \begin{bmatrix} \rho_{aa} \\ \rho_{bb} \\ \rho_{dd} \\ \rho_{ad} \\ \rho_{da} \\ \rho_{bd} \\ \rho_{db} \\ \rho_{ab} \\ \rho_{ba} \end{bmatrix}, \quad \widetilde{D} = \alpha r_a \rho \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

We assume that the atoms in state $\rho_a(0) = |d\rangle\langle d|$ are injected into the cavity with rate r_a. The solution of (6) is given by $\tilde{R}_i = \alpha r_a \tilde{M}_{i,3}^{-1}\rho = Q_i\rho$ $(i = 1, 2, \ldots, 9)$. Here, $\widetilde{M}_{i,3}^{-1}$ means the elements in ith row and $3rd$ column of the matrix $\widetilde{M}^{-1}$ (9 by 9 matrix, the inverse of the matrix $\widetilde{M}$). If we arrange those 9 numbers Q_i into a 3×3 matrix

$$Q = \begin{bmatrix} Q_1 & Q_9 & Q_5 \\ Q_8 & Q_2 & Q_7 \\ Q_4 & Q_6 & Q_3 \end{bmatrix}$$

we will get the coefficients in master equation can be written as the elements of a matrix F which is the multiplication of the inverse matrix M^{-1} and matrix Q

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = M^{-1}Q. \tag{8}$$

Finally, we can derive the following master equation for the cavity modes of the system

$$\begin{aligned} \dot{\rho} = {} & -[g_1^2 F_{11}(a_1 a_1^\dagger \rho - a_1^\dagger \rho a_1) + g_1 g_2 F_{12}(a_1 a_2^\dagger \rho - a_2^\dagger \rho a_1) \\ & + g_1 g_3 F_{13}(a_1 a_3 \rho - a_3 \rho a_1) + g_1 g_2 F_{21}(a_2 a_1^\dagger \rho - a_1^\dagger \rho a_2) \\ & + g_2^2 F_{22}(a_2 a_2^\dagger \rho - a_2^\dagger \rho a_2) + g_2 g_3 F_{23}(a_2 a_3 \rho - a_3 \rho a_2) \\ & - g_1 g_3 F_{31}^*(a_3 \rho a_1 - \rho a_1 a_3) - g_2 g_3 F_{32}^*(a_3 \rho a_2 - \rho a_2 a_3) \end{aligned}$$

$$- g_3^2 F_{33}^*(a_3\rho a_3^\dagger - \rho a_3^\dagger a_3) + \varepsilon(\rho a_1^\dagger a_2^\dagger - a_1^\dagger a_2^\dagger \rho) + h.c.]$$
$$+ \sum_{i=1,2,3} \kappa_i (2a_i\rho a_i^+ - a_i^+ a_i \rho - \rho a_i^+ a_i) \tag{9}$$

where κ_i is the cavity damping constant for mode i. We will discuss the entanglement of the two-mode fields from the master equation.

3 Multi-mode Entanglement

In order to detect the entanglement, we briefly review the positivity of partial transpose (PPT) criterion for CV system [26, 33, 34]. Consider n-mode states with annihilation and creation operators a_j and $a_j^\dagger$ $(j = 1, 2, \dots, n)$ obeying the standard boson commutation relations $[a_j, a_k^\dagger] = \delta_{jk}$. The Hermitian operators q_j and p_j are defined by $q_j = (a_j + a_j^\dagger)$, $p_j = -i(a_j - a_j^\dagger)$. It is convenient to arrange the Hermitian operators q_j and p_j into a $2n$-component vector as $\vec{x} = [q_1, \dots, q_n, p_1, \dots, p_n]$. The commutation relations can be collectively written as $[\vec{x}_i, \vec{x}_j] = 2i\beta_{ij}$ [35], where the $2n \times 2n$ matrix β is given in a block form by $\beta = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ with I is an unit matrix.

From the vector $\vec{x}$, we can define the $2n \times 2n$ real covariance matrix V for the state ρ by [35]

$$V_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(\rho\{\vec{x}_\mu, \vec{x}_\nu\}), \tag{10}$$

where a pair of curly bracket denote the anti-commutation relation between the operators. It can also be written in terms of $n \times n$ blocks

$$\begin{aligned} V &= \begin{bmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{bmatrix}, \\ (V_1)_{jk} &= \langle q_j q_k \rangle, \\ (V_2)_{jk} &= \frac{1}{2}\langle \{q_j, p_k\} \rangle, \\ (V_3)_{jk} &= \langle p_j p_k \rangle, \quad j, k = 1, 2, \dots, n. \end{aligned} \tag{11}$$

In general, the covariance matrix is defined as $V_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(\rho\{\triangle\vec{x}_\mu, \triangle\vec{x}_\nu\})$. We assume that all the initial states of three modes are vacuum states, then one can easily examine that all the mean values of the operators $\langle \vec{x}_\mu \rangle$ would be zero not only in the initial states but also during the time evolution for the form of master equation (2). Therefore, $\triangle\vec{x}_\mu = \vec{x}_\mu - \langle \vec{x}_\mu \rangle$ will be simplified to $\triangle\vec{x}_\mu = \vec{x}_\mu$, and the covariance matrix can be defined as (10).

In order to represent a physical state, the covariance matrix must obey the Robertson-Schrodinger uncertainty principle [36, 37],

$$V + i\beta \geq 0. \tag{12}$$

Now, we consider the partial transpose of the covariance matrix V. The partial transpose operation is interpreted as a mirror reflection in the Wigner phase space so that partial transpose matrix $\widetilde{V}$ can be obtained from V just by taking p_j in $-p_j$ for a given mode [8, 26].

If the transposed mode is separable from the others, then the partially transposed matrix $\widetilde{V}$ still obey the inequality in (12). In order to see whether the transposed matrix $\widetilde{V}$ satisfies the inequality in (12) (i.e. $\widetilde{V} + i\beta \geq 0$.), we can examine if all the symplectic eigenvalues of the transposed matrix are larger than 1. These symplectic eigenvalues can be computed as the square roots of the ordinary eigenvalues of $-(\beta\widetilde{V})^2$. Obviously, the inequality (12) demands that the square roots of all the eigenvalues of $-(\beta\widetilde{V})^2$ be larger than 1. So, if the smallest eigenvalue $\widetilde{\nu}_{\min}$ is smaller than 1, the transposed mode is then inseparable [8]. The violation of the inequality is a sufficient condition for the existence of entanglement between the transposed mode and the remaining modes.

Although the PPT criterion has just been proved to be sufficient and necessary for Gaussian states in Simon's paper, according to recent work, we can see that the PPT criterion is at least a sufficient condition for entanglement no matter the states are Gaussian or non-Gaussian. We recall Shchukin's work [33], where Shchukin and Vogel (SV) have introduced an elegant and unify approach based on the PPT requirement no matter whether the states are Gaussian or non-Gaussian, pure or mixed. SV show that Simon's PPT criterion is just a special case of SV criterion. The violation of Simon's PPT criterion will definitely lead to the violation of the SV criterion. So, the violation of PPT will be a sufficient (although maybe not necessary) conditions for entanglement, no matter the states are Gaussian or non-Gaussian, pure or mixed. As Adesso et al. concluded, the PPT criterion is in principle to detect all CV states, pure or mixed, Gaussian or non-Gaussian.

For tripartite, if all the three partial transposition are nonpositive, the mixed states will be a class of "fully inseparable states" [39], also being called "free tripartite entanglement" [27].

Now, we begin to prove that the existence of "genuine tripartite entanglement" in our system. To calculate the eigenvalues of matrix V, we rewrite the covariance matrix V in term of a_j and $a_j^\dagger$. Then, all the elements of the variance matrix are composed of a series of mean values, $\langle a_j a_k\rangle$, $\langle a_j a_k^\dagger\rangle$, $\langle a_j^\dagger a_k\rangle$, and $\langle a_j^\dagger a_k^\dagger\rangle$. These mean values can be easily calculated from the master equation (9) in using the relation

$$\frac{d}{dt}\langle\hat{O}\rangle = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{O}\right). \tag{13}$$

Substituting the master equation into the equation above, one can obtain $\frac{d}{dt}\langle\vec{E}\rangle = M_1\langle\vec{E}\rangle + M_2$, where $\langle\vec{E}\rangle$ is a vector containing all the mean values of various moments such as $\langle a_j a_k\rangle$, $\langle a_j a_k^\dagger\rangle$, $\langle a_j^\dagger a_k\rangle$, $\langle a_j^\dagger a_k^\dagger\rangle \ldots$ while M_1 is a constant matrix whose elements are determined by the coefficients in the master equation and M_2 is a vector whose elements are constants. Solving this differential equation, one can obtain the mean values $\langle a_j a_k\rangle$, $\langle a_j a_k^\dagger\rangle$, $\langle a_j^\dagger a_k\rangle$, and $\langle a_j^\dagger a_k^\dagger\rangle$. Substitute them into the covariance matrix (11), then we can calculate the smallest eigenvalue, obtain its time evolution, and check whether the transposed mode is separable. Because the expression of matrix M and the analytical solution of the differential equation is tedious and meaningless, we only give the method of the calculation here, and show our numerical results.

We are now in a position to present our numerical results for the smallest eigenvalues and the mean photon numbers. In the following, we assume that the initial states for all the three modes are vacuum states.

In Fig. 2, we plot the smallest eigenvalues of the bipartite partially transposed covariance matrix and the photon numbers of the three modes as the function of time (in terms of gt). In this figure, the OPO is not considered, so that the parameter ε is zero. The red (solid), green (dashed) and blue (dash-dotted) lines represent the bipartite partial transposed matrix of modes 1–2, 1–3 and 2–3 respectively. We just call them E_{1-2}, E_{2-3}, E_{1-3}. From Fig. 2(a) we see that bipartite entanglement exist in modes 1–3 and 2–3 (the smallest eigenvalues

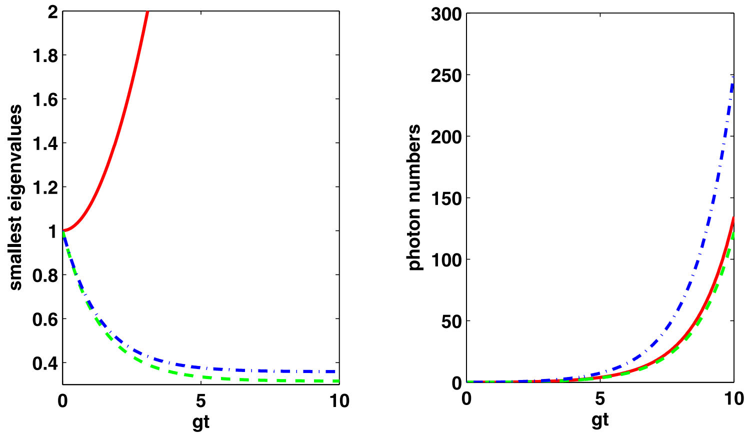

Fig. 2 Time evolution of entanglement and mean photon numbers. In the *left picture*, the *red* (*solid*), *green* (*dashed*) and *blue* (*dash-dotted*) lines represent the smallest eigenvalues of the partially transposed covariance matrix for modes 1–2, 1–3, and 2–3, respectively. In the *right picture*, the *red* (*solid*), *green* (*dashed*) and *blue* (*dash-dotted*) lines represent the photon numbers of the modes 1, 2, and 3, respectively. The parameters are $g_1 = g_2 = g_3 = g$, $\Omega_{ac} = 10.5g$, $\Omega_{bd} = 10g$, $\gamma = 0.1g$, $r_a = 1$, $\varphi_1 = \varphi_2 = \pi/2$, $\varepsilon = 0$, and $\kappa_1 = \kappa_2 = \kappa_3 = 0.01g$

can be smaller than 1), but there is no entanglement between modes 1 and 2 (the red line is always bigger than 1). This is in agreement with our analysis in Sect. 2. Further more, we can also explain the entanglement by analyzing our master equation. In order to see the entanglement structure, we omit the damping terms and assume $\Omega_1(\Omega_2) \gg \gamma$. Then, there are only nine terms in the master equation (in fact there are 18 terms in the master equation if the conjugation are included) whose coefficients are determined by the elements of matrix F. We just want to compare the relative magnitudes of these coefficients, so we multiply all these coefficients by a constant $\mathrm{Det}(M)\,\mathrm{Det}(\widetilde{M})$, then under a special case $\varphi_1 = \varphi_2 = \frac{\pi}{2}$, a relatively simple expression of matrix F can be derived.

$$F \propto \begin{bmatrix} 0 & 0 & -2\Omega_1({\Omega_1}^2+{\Omega_2}^2)^3 \\ 0 & 0 & -2\Omega_2({\Omega_1}^2+{\Omega_2}^2)^3 \\ 2\Omega_1({\Omega_1}^2+{\Omega_2}^2)^3 & 2\Omega_2({\Omega_1}^2+{\Omega_2}^2)^3 & 0 \end{bmatrix} \tag{14}$$

we can see that only four terms i.e. $g_1g_3F_{13}(a_1a_3\rho - a_3\rho a_1)$, $g_2g_3F_{23}(a_2a_3\rho - a_3\rho a_2)$, $g_1g_3F_{31}(a_3\rho a_1 - \rho a_1a_3)$ and $g_2g_3F_{23}(a_3\rho a_2 - \rho a_2a_3)$ are nonzero, all the other terms are approximately equal to zero. Or, to be more precise, the other terms are negligible when compared to these four terms. Thus, the master equation can be reduced into

$$\begin{aligned} \dot{\rho} \approx -[&g_1g_3F_{13}(a_1a_3\rho - a_3\rho a_1) + g_2g_3F_{23}(a_2a_3\rho - a_3\rho a_2) \\ &- g_1g_3F_{31}^*(a_3\rho a_1 - \rho a_1a_3) - g_2g_3F_{32}^*(a_3\rho a_2 - \rho a_2a_3) \\ &+ \varepsilon(\rho a_1^\dagger a_2^\dagger - a_1^\dagger a_2^\dagger\rho) + h.c.] + \sum_{i=1,2,3}\kappa_i(2a_i\rho a_i^+ - a_i^+a_i\rho - \rho a_i^+a_i) \end{aligned} \tag{15}$$

and the effective Hamiltonian can be written as

$$H_{eff} = g_1g_3F_{13}a_1a_3 + g_2g_3F_{23}a_2a_3 + \varepsilon a_1a_2 + h.c. \tag{16}$$

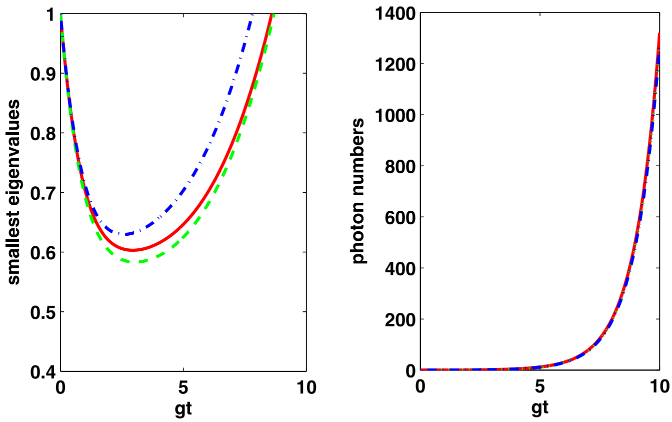

Fig. 3 Time evolution of entanglement and mean photon numbers with OPO in the cavity. In the *left picture*, the *red* (*solid*), *green* (*dashed*) and *blue* (*dash-dotted*) lines represent the smallest eigenvalues of the partially transposed covariance matrix for modes 1–2, 1–3, and 2–3, respectively. In the *right picture*, the *red* (*solid*), *green* (*dashed*) and *blue* (*dash-dotted*) lines represent the photon numbers of the modes 1, 2, and 3, respectively. All the parameters are the same with Fig. 2 except $\varepsilon = 0.245g$

Now, we can see clearly that it describes three parametric oscillators, and we can understand why the entanglement exist. It describe the process $|a\rangle \xrightarrow{a_1^\dagger} |c\rangle \xrightarrow{a_3^\dagger} |d\rangle$, $|b\rangle \xrightarrow{a_2^\dagger} |c\rangle \xrightarrow{a_3^\dagger} |d\rangle$, and the OPO $a_1^\dagger a_2^\dagger$ (see Fig. 1). Moreover, the state with parametric oscillator (16) is of Gaussian states form. It has been proved that PPT criterion for Gaussian states is a sufficient and necessary for CV system. So, the $\widetilde{\nu}_{\min}$ is a measure for the state with Hamiltonian (16). Besides, we can see the photon numbers are increase as time evolution, so our scheme can be used as the photon amplifier. For the parameters we choose in Fig. 2, g/γ is about 10, and g/κ is about 100. This kind of coupling to dissipation ratio can be provided by many systems. Even the g/κ ratio can be also satisfied by toroidal microcavity [40].

We can not neglect the fact that when the OPO is not considered, there is no entanglement between mode 1 and 2 as Fig. 2 shows. (In fact, we can not definitely know that entanglement do not exist between modes 1 and 2, just from the fact that the smallest eigenvalue is larger than one. Because the PPT criterion is not a necessary condition for entanglement.) Next, we will take a look at what the OPO will bring to us when it works. In Fig. 3, we plot bipartite entanglement of every two modes and the photon number of the three modes as the function of time (in terms of gt) again under the condition that the OPO works ($\varepsilon = 0.245g$). It shows that entanglement are produced between every two modes of all the three modes and the photon numbers of the three modes are amplified more significantly. The increasing of photon number result from two reasons. The one is because of the pumping of classical field Ω_1 and Ω_2. The other is because of OPO's working. However, one may surprisingly find that after a period of evolution every bipartite entanglement disappear just while introducing the OPO with $\varepsilon = 0.245g$. Does the OPO destroy every bipartite entanglement? The answer is not. To easily understand the entanglement structure, we still employ the Hamiltonian (16) and try to explain it. When $\varepsilon = 0$, for initial state $|0,0,0\rangle$, the evolution of the state if we only consider the mode 1 and 3, then at $t = 0$, $\Psi(t) = \sum_n \frac{1}{n!}(H_{eff})^n|0,0,0\rangle$. Thus, one can see that the reduced density matrix ρ_{13} and ρ_{23} contain the components of entanglement $(c_1|00\rangle + c_2|11\rangle + c_3|22\rangle + \cdots)(c_1\langle 00| + c_2\langle 11| + c_3\langle 22| + \cdots)$ while ρ_{12} only have $(c_1|01\rangle + c_2|10\rangle)(c_1\langle 01| + c_2\langle 10|) + \cdots$. As we know, PPT is a good measure for the OPO

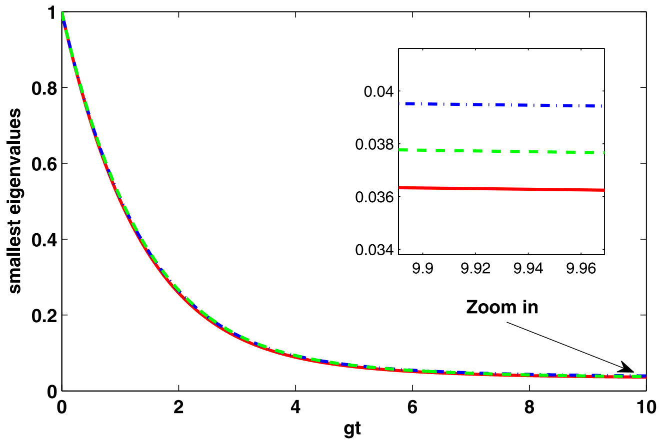

Fig. 4 Time evolution of the smallest eigenvalues of the partially transposed covariance matrix. In this figure, one mode is transposed, while the other two modes are not transposed. The *red* (*solid*), *green* (*dashed*) and *blue* (*dash-dotted*) lines represent the partially transposed mode are mode 1–23 (mode 1 is transposed while mode 2 and 3 are not transposed), 2–13 and 3–12 respectively. The parameters are the same with Fig. 3

with $c_1|00\rangle + c_2|11\rangle + c_3|22\rangle + \cdots$ but fail to measure the state with $c_1|01\rangle + c_2|10\rangle$. Consequently, in Fig. 2, we only can observe the entanglement of 1–3 and 2–3. But for nonzero ε [with (16)], the states may evolve into $|000\rangle$, $|222\rangle$, $|444\rangle \ldots$. So the total state at time t may contain the superposition of these kinds of states, like $c_1|000\rangle + c_2|222\rangle + c_3|444\rangle + \cdots$ (like GHZ state). If the component of this kind of states is obviously larger than other kinds of components, any bipartite entanglement will be zero. We know that any reduced bipartite has no entanglement but we have genuine tripartite entanglement in GHZ state, this may explain why we can not detect entanglement in our states. When the system is in a mixed state, it become more complicated, but the entanglement structure keep unchanged. That is to say, above explanation might be helpful for understand Figs. 2 and 3.

For a tripartite system, there will be six kinds of partial transpositions to describe the separability of the whole system. We have already study three of them E_{1-2}, E_{2-3} and E_{3-1}, and we are going to investigate the remaining three ones. With the same group of parameters, in Fig. 4, we show the smallest eigenvalues for all the three partially transposed covariance matrix. We just call them E_{1-23} (mode 1 is partially transposed, often used to test the separability between mode 1 and the other two modes), E_{2-13} (mode 2 is partially transposed) and E_{3-12} (mode 3 is partially transposed) respectively. We observe that E_{1-23}, E_{2-13} and E_{3-12} are all smaller than 1 as time evolution although E_{1-2}, E_{2-3} and E_{3-1} can be zero in Fig 3, which prove that our above explanation is correct. Since all E_{1-23}, E_{2-13} and E_{3-12} are smaller than 1, we can safely say that a genuine n-partite entanglement is present in the system, because Shchukin and Vogel have shown us that if all the partial transposition are nonpositive (corresponding to all the smallest eigenvalues are smaller than 1), it is then said that genuine n-partite entanglement is present in the system [38]. Moreover, Dur and Cirac also indicate that, if all the three partial transposition are nonpositive, the mixed states will be a class of "fully inseparable states" [39]. Thus, we have proved the existence of the genuine tripartite entanglement in our scheme.

4 Conclusion

In summary, we have proposed a scheme for generating genuine three-mode entangled light in cavity with Y-type atoms. It is proved that entanglement exists in every mode with the others and all the three modes are inseparable from the remain parts. We analyze the master equation and show the mechanism of entanglement generation. In addition, we have proved the genuine three-mode entanglement is generated in the system.

During the review process of this paper, we notice that one similar theoretical research [41] consider the generation of three-mode entanglement in a Y-type atomic system where they showed that this kind of atomic system can generate three-mode entanglement. Fortunately, we employ different entanglement criterion with their and analyse the entanglement forms. Furthermore, we introduce a OPO and infer that a state containing GHZ component can be generated.

Acknowledgements We would like to acknowledge the support by NSFC under Grant No. 10774020 and by SRF for ROCS of SEM.

References

1. Braunstein, S.L., Van Lood, P.: Rev. Mod. Phys. **77**, 513 (2005)
2. Serafini, A., Paternostro, M., Kim, M.S., Bose, S.: Phys. Rev. A **73**, 022312 (2006)
3. Van Loock, P., Braunstein, S.L.: Phys. Rev. Lett. **84**, 3482 (2000)
4. Van Loock, P., Braunstein, S.L.: Phys. Rev. Lett. **87**, 247901 (2001)
5. Zhang, J., Xie, C., Peng, K.: Phys. Rev. A **66**, 02318 (2002)
6. Aoki, T., Takei, N., Yonezawa, H., et al.: Phys. Rev. Lett. **91**, 080404 (2003)
7. Yonezawa, H., Aokl, T., Furusawa, A.: Nat. Lond. **431**, 430 (2004)
8. Cassemiro, K.N., Villar, A.S.: Phys. Rev. A **77**, 022311 (2008)
9. Xiong, H., Scully, M.O., Zubairy, M.S.: Phys. Rev. Lett. **94**, 023601 (2005)
10. Tan, H.T., Zhu, S.Y., Zubairy, M.S.: Phys. Rev. A **72**, 022305 (2005)
11. Li, G.X., Tan, H.T., Macovei, M.: Phys. Rev. A **76**, 053827 (2007)
12. Kiffner, M., Zubairy, M.S., Evers, J., Keitel, C.H.: Phys. Rev. A **75**, 033816 (2007)
13. Tesfa, S.: J. Phys. B **41**, 055503 (2008)
14. Zhou, L., Xiong, H., Zubairy, M.S.: Phys. Rev. A **74**, 022321 (2006)
15. Zhou, L., Xiong, H.: J. Phys. B **41**, 025501 (2008)
16. Zhou, L., Ma, Y.H., Zhao, X.Y.: J. Phys. B **41**, 215501 (2008)
17. Ma, Y.H., Mu, Q.X., Yang, G.H., Zhou, L.: J. Phys. B **41**, 215502 (2008)
18. Alebachew, E.: Phys. Rev. A **76**, 023808 (2007)
19. Qamar, S., Xiong, H., Zubairy, M.S.: Phys. Rev. A **75**, 062305 (2007)
20. Zhang, Y., Wang, H., Li, X.Y., Jing, J.T., Xie, C.D., Peng, K.C.: Phys. Rev. A **62**, 023813 (2000)
21. Simon, C., Bouwmeester, D.: Phys. Rev. Lett. **91**, 053601 (2003)
22. Guo, J., Zou, H.X., et al.: Phys. Rev. A **71**, 034305 (2005)
23. Bradley, A.S., Olsen, M.K.: Phys. Rev. A **72**, 053805 (2005)
24. Pennarun, C., Bradley, A.S., Olsen, M.K.: Phys. Rev. A **76**, 063812 (2007)
25. Zhao, X.Y., Ma, Y.H., Zhou, L.: Opt. Commun. **282**, 1593 (2009)
26. Simon, R.: Phys. Rev. Lett. **84**, 2726 (2000)
27. Yu, C.S., Song, H.S.: Phys. Lett. A **330**, 377 (2004)
28. Agarwal, G.S., Harshawardhan, W.: Phys. Rev. Lett. **77**, 1039 (1996)
29. Arun, R.: Phys. Rev. A **77**, 033820 (2008)
30. Hou, B.P., et al.: Phys. Rev. A **69**, 053805 (2004)
31. Gao, J.Y., Yang, S.H., Wang, D., et al.: Phys. Rev. A **61**, 023401 (2000)
32. Scully, M.O., Zubairy, M.S.: Quantum Optics. Cambridge University Press, Cambridge (1997)
33. Shchukin, E., Vogel, W.: Phys. Rev. Lett. **95**, 230502 (2005)
34. Adesso, G., Illuminati, F.: J. Phys. A **40**, 7821 (2007)
35. Simon, R., Mukunda, N., Dutta, B.: Phys. Rev. A **49**, 1567 (1994)
36. Schrodinger, E., et al.: Phys. Rev. **24**, 296 (1930)
37. Robertson, H.P.: Phys. Rev. **46**, 794 (1934)

38. Shchukin, E., Vogel, W.: Phys. Rev. A **74**, 030302(R) (2006)
39. Dur, W., Cirac, J.I.: Phys. Rev. A **61**, 042314 (2000)
40. Spillane, S.M., Kippenberg, T.J., Vahala, K.J.: Phys. Rev. A **71**, 013817 (2005)
41. Lü, X.-Y., et al.: Phys. Rev. A **80**, 032305 (2009)